TRINITY COLLEGE

Year 12 Mathematics Specialist 3/4
Test 1 2022

Section 1 Calculator Free

Complex Numbers

STUDENT'S NAME Solutions

DATE: Monday 28 February **TIME:** 20 minutes **MARKS**: 19

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (4 marks)

The function $f(x) = x^4 - x^3 + ax^2 + bx - 18$ has a root at $x = -2$ and has a remainder of -20 when divided by $(x-1)$.

Determine the values of a and b where $a, b \in \mathbb{Z}$.

$f(-2) \Rightarrow 16 + 8 + 4a - 2b - 18 = 0$ ✓ uses $x=-2$

$\Rightarrow 4a - 2b + 6 = 0$

$\Rightarrow 2a - b + 3 = 0$ — (1)

$f(1) \Rightarrow 1 - 1 + a + b - 18 = -20$ ✓ uses $x = 1$

$a + b + 2 = 0$ — (2)

Adding (1) + (2) yields

$\Rightarrow 3a + 5 = 0$ ✓ solves for a

$\Rightarrow a = -\frac{5}{3}$

$\Rightarrow b = -\frac{1}{3}$ ✓ solves for b

2. (7 marks)

Consider the polynomial function $f(z) = z^4 + 7z^2 + 12$

(a) Show that $z - 2i$ is a factor of $f(z)$ [2]

$$f(2i) = 16i^4 + 7.4i^2 + 12$$
$$= 16 - 28 + 12$$
$$= 0$$

✓ subs in $2i$

✓ shows all values sum to 0

(b) State another factor of $f(z)$ [1]

$(z + 2i)$

✓ answer

(c) Hence, or otherwise, solve $f(z) = 0$ [4]

So $(z - 2i)(z + 2i)(az^2 + bz + c) = z^4 + 0z^3 + 7z^2 + 0z + 12$

$\Rightarrow (z^2 + 4)(az^2 + bz + c) = z^4 + 7z^2 + 12$

by inspection $a = 1$
$c = 3$
$b = 0$

✓ states factor (z^2+4)

✓ solves for second quadratic

We now have

$$f(z) = (z^2+4)(z^2+3)$$

✓ factorises or states 2 solns

Solving $(z^2+4)(z^2+3) = 0$

$\Rightarrow z = \pm 2i, \pm\sqrt{3}\,i$

✓ states all solns

✓ z^2

✓✓ mentions 1st quadrant and not in domain

3. (8 marks)

Consider the locus of points defined for $\left\{z : z \in \mathbb{C}, \frac{\pi}{2} \le \arg\left(z^2\right) < \pi\right\}$

(a) Show that $z = 2 + i$ is **not** in the locus. Explain. [3]

$z^2 = (2+i)^2$

$= 4 + 4i + i^2$

$= 3 + 4i$

$Arg(z^2) = \tan\left(\frac{4}{3}\right)$

$3+4i$ is in the 1st quadrant.

$\therefore \arg(z^2) \le \frac{\pi}{2}$

and is not $\ge \pi/2$

$\therefore$ it is not in the locus

(b) Show that $z = 1 + i$ is in the locus. [2]

$z^2 = (1+i)^2$

$= 1 + 2i + i^2$

$= 2i$

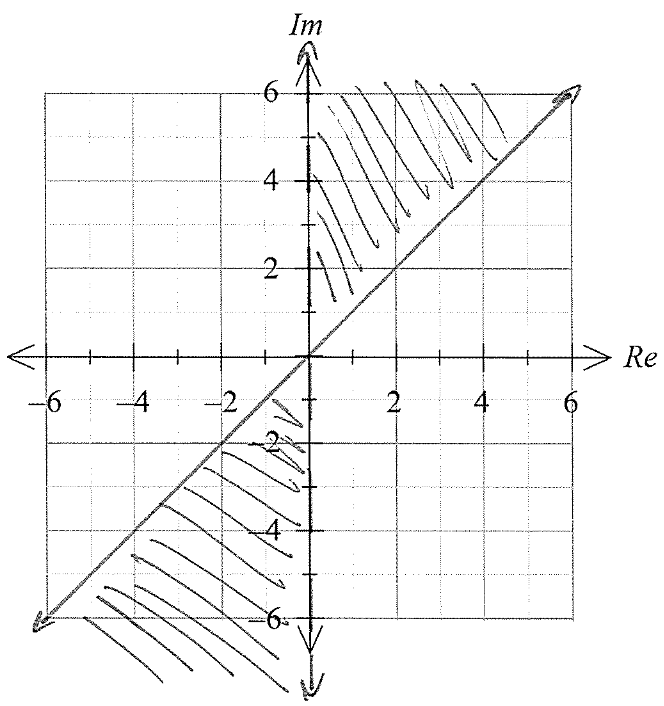
$Arg(z^2) = \frac{\pi}{2}$

✓ z^2

✓ $\pi/2$

This is on the boundary of the locus.

(c) On the Argand plane below, sketch the locus of points of z. [3]



✓ $\pi/4 < \arg z < \frac{\pi}{2}$

✓ correct boundary lines

✓ $-\frac{3\pi}{4} < \arg(z) < -\frac{\pi}{2}$

TRINITY COLLEGE

Year 12 Mathematics Specialist 3/4
Test 1 2022

Section 2 Calculator Assumed
Complex Numbers

STUDENT'S NAME ______________________________

DATE: Monday 28 February **TIME:** 30 minutes **MARKS**: 31

INSTRUCTIONS:

Standard Items:	Pens, pencils, drawing templates, eraser
Special Items:	Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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4. (6 marks)

Given the complex numbers $w = 2 - 2i$ and $u = 3cis\dfrac{\pi}{4}$, determine:

(a) $\arg\left(\dfrac{2i - \overline{w}}{u}\right)$ $= \arg(2i - \overline{w}) - \arg(u)$ [3]

✓ simplifies argument

$= \arg(2i - (2+2i)) - \arg(u)$

$= \arg(-2) - \arg(u)$ ✓ simplifies numerator

$= \pi - \dfrac{\pi}{4}$ ✓ answer

$= \dfrac{3\pi}{4}$

(b) $|u^2 w^2|$ $= |3^2| \cdot |2^2 + (-2)^2|$ [3]

$= 9 \times 8$ ✓ $|u^2|$

$= 72$ ✓ $|w^2|$

✓ answer

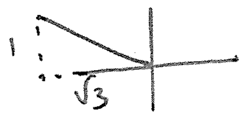
5. (7 marks)

Consider the complex equation $z^5 + 16\sqrt{3} - 16i = 0$.

(a) Solve the equation giving exact solutions in the form $r\,cis\,\theta$ where $-\pi < \theta \le \pi$. [4]

$$z^5 = -16\sqrt{3} + 16i$$

$$= 32\,cis\left(\frac{5\pi}{6} + 2\pi k\right)$$



So $z_k = 32^{1/5}\,cis\left(\frac{5\pi}{30} + \frac{12\pi k}{30}\right)$

✓ z^5 in polar form

So $z_0 = 2\,cis\,\frac{5\pi}{30}$

✓ De Moivres

$z_1 = 2\,cis\,\frac{17\pi}{30}$

✓ 2 correct solns

$z_2 = 2\,cis\,\frac{29\pi}{30}$

✓ All correct solns in standard domain

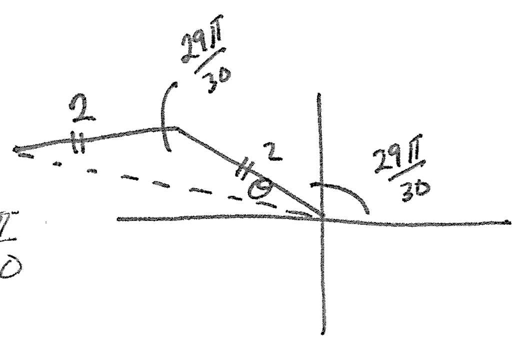
$z_4 = 2\,cis\,\frac{-19\pi}{30}$

$z_5 = 2\,cis\,\frac{-7\pi}{30}$

Let w be the solution to $z^5 + 16\sqrt{3} - 16i = 0$ with the greatest argument.

(b) Determine the exact value for $\arg(w-2)$ [3]

$w = 2\,cis\,\frac{29\pi}{30}$



θ is angle in isosceles $\triangle$

$\Rightarrow \theta = \frac{1}{2}\left(\pi - \frac{29\pi}{30}\right) = \frac{\pi}{60}$

$\therefore \arg(w-2) = \frac{29\pi}{30} - \frac{\pi}{60}$

$= \frac{59\pi}{60}$

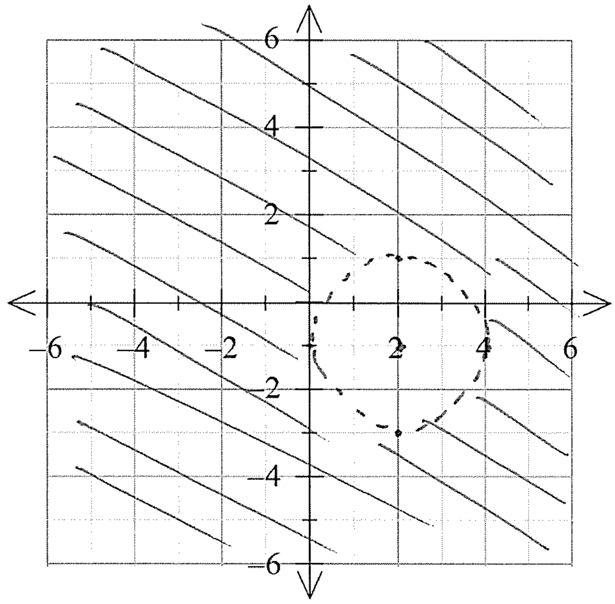
✓ w

✓ diagram

6. (10 marks)

(a) On the Argand planes below, sketch the locus of the complex number $z = x + iy$ given by:

(i) $\{z : z \in \mathbb{C}, |z - 2 + i| > 2\}$ $\Rightarrow$ $|z - (2 - i)| > 2$ [3]

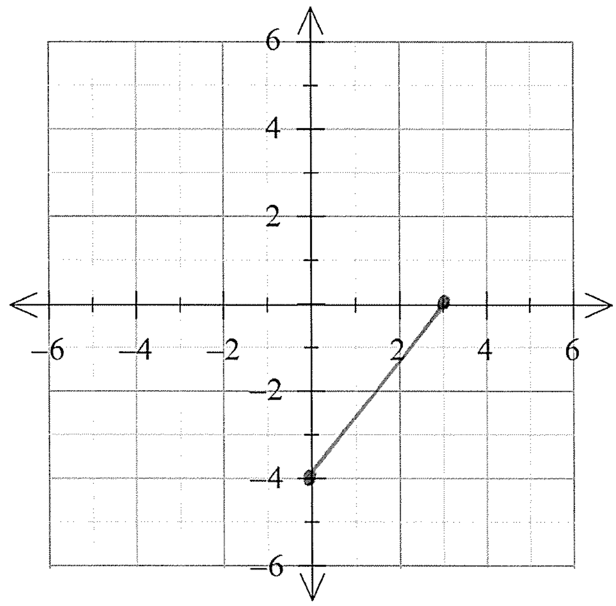


✓ circle centre $2 - i$

✓ dashed radius $r = 2$

✓ outside region

(ii) $\{z : z \in \mathbb{C}, |z - 3| + |z + 4i| = 5\}$ $\Rightarrow$ $|z - (3)| + |z - (-4i)| = 5$ [3]

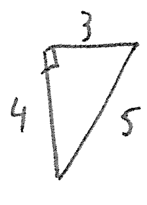


✓ 3

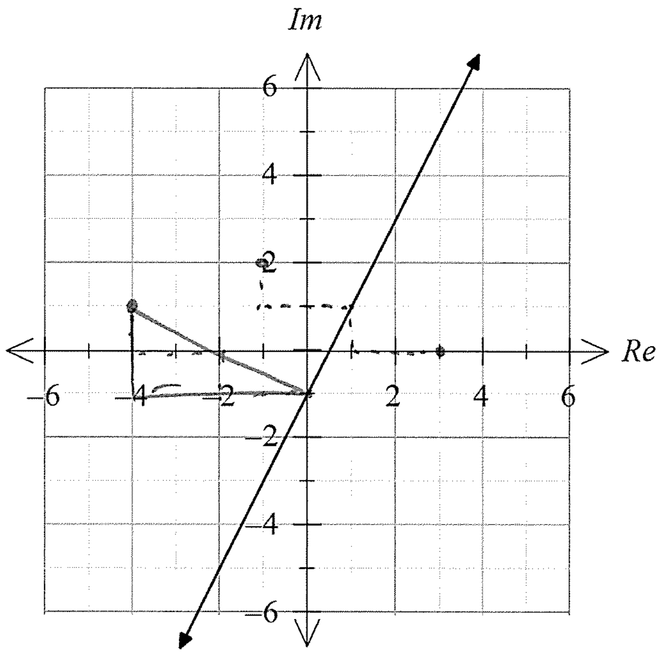
✓ $-4i$

✓ line segment between pts

"distance from 3 + distance from $-4i$ = 5"



(b) A sketch of the locus of a complex number $z = x + iy$ is shown below.



(i) The equation is of the form $|z - w| = |z - 3|$ where $w \in \mathbb{C}$. Determine the value of w. [1]

$w = -1 + 2i$ ✓ answer

(ii) Determine the minimum value for $|z + 4 - i|$ as an exact value. [3]

min dist $|z - (-4 + i)|$

Using pythagoras ✓ diagram

min dist $= \sqrt{2^2 + 4^2}$ ✓ pythagoras

$= \sqrt{20}$ ✓ answer as exact value

$= 2\sqrt{5}$

7. (8 marks)

Consider the locus of the complex number $z = x + iy$ given by $(1-i)z + (1+i)\overline{z} = 4$.

(a) Show that $z = 2i$ is in the locus. [2]

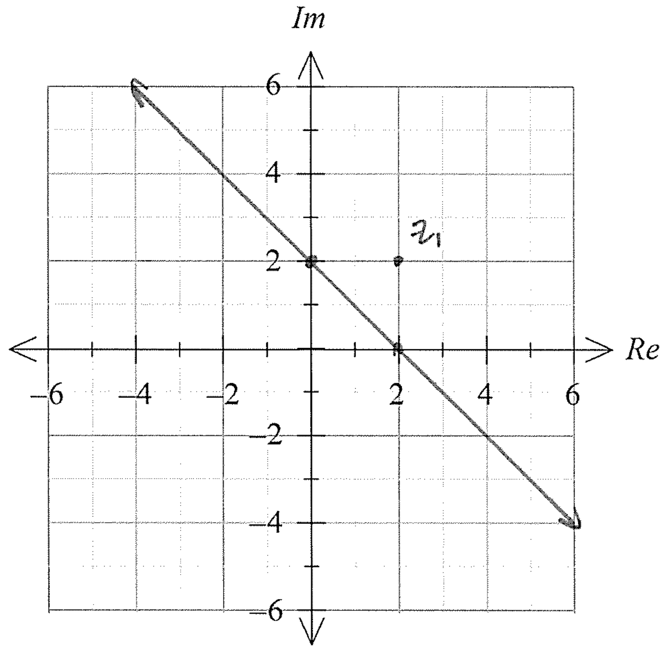
$(1-i)2i + (1+i)(-2i)$ ✓ sub

$= \cancel{2i} + 2 - \cancel{2i} + 2$ ✓ answer

$= 4$

The locus forms a line.

(b) Hence, or otherwise, sketch the locus on the Argand diagram below. [2]



✓ line

✓ one other pt ie 2

Let $z = x + iy$

$\Rightarrow (1-i)(x+iy) + (1+i)(x-iy) = 4$

$\Rightarrow x + \cancel{iy} - \cancel{xi} + y + x - \cancel{iy} + \cancel{xi} + y = 4$

$\Rightarrow x + y = 2$

Let $z_1 = 2 + 2i$ be a point in the complex plane

(c) If the reflection of z_1 about the line in part (b) is z_2, calculate the value of $\overline{z}_1(1+i) + z_2(1-i)$. [4]

from part (b), $z_2 = 0$ ✓ draws z_1

✓ z_2

So $(2-2i)(1+i) + 0(1-i)$

$= 2 + 2i - 2i + 2$ ✓ Subs

$= 4$ ✓ answer